Unit - 4

Func^n of a Complex Variable

(1) **Complex No. :-** A no. of the form $x+iy$, where $\sqrt{-1} = i$ and (x, y) both are real numbers is called a complex no. It is also defined as an ordered pair of (x, y). A complex no. is usually denoted by $z = x+iy$ where, x is real part and y is imaginary part of z.

Equality of two complex No.'s :- Two no.'s $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if $x_1 = x_2$ and $y_1 = y_2$.

Addition of Complex No. :- If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then,
$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication of two complex No. :-

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Subtraction of two complex No. :-

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

Then,
$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

Division of two complex No. :-

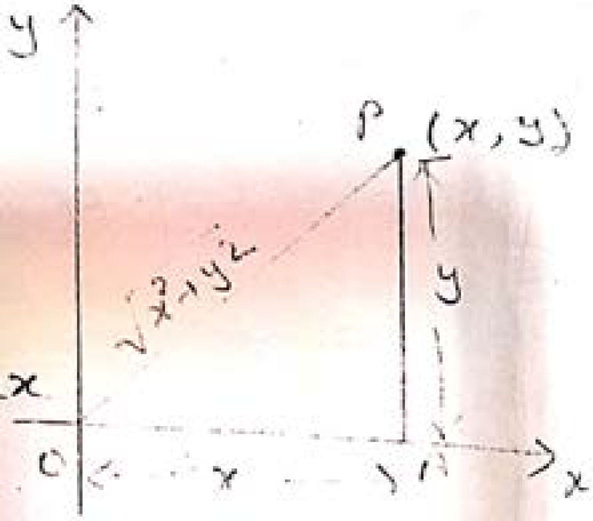
A complex no. $(a+ib)$ is divisable by a complex no. $(c+id)$ if $\exists$ a complex no. $(x+iy)$ s.t.

[$a = b \cdot c$, b/a

$$(a+ib) = (c+id)(x+iy)$$

modulus of a complex No. :-

If $z = x+iy$ is any complex no. then the non negative real no. $\sqrt{x^2+y^2}$ is called the modulus of the complex no. and denoted by $|z|$



i.e. $z = x+iy$
$$|z| = \sqrt{x^2+y^2}$$

Geometrically, modulus of a complex no. distance b/w origin and the co-ordinate of that complex no.

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Properties :-

(i) $|z| \geq Re(z)$, $|z| \geq Im(z)$

(ii) $|z_1 . z_2| = |z_1||z_2|$

(iii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

Conjugate of a Complex No :- If $z = x+iy$ is any complex no. then $(x-iy)$ is called conjugate of z and denoted by $\bar{z}$.

properties :-

(i) $|z| = |\bar{z}|$

(ii) $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$

(iii) $\overline{(\bar{z})} = z$

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

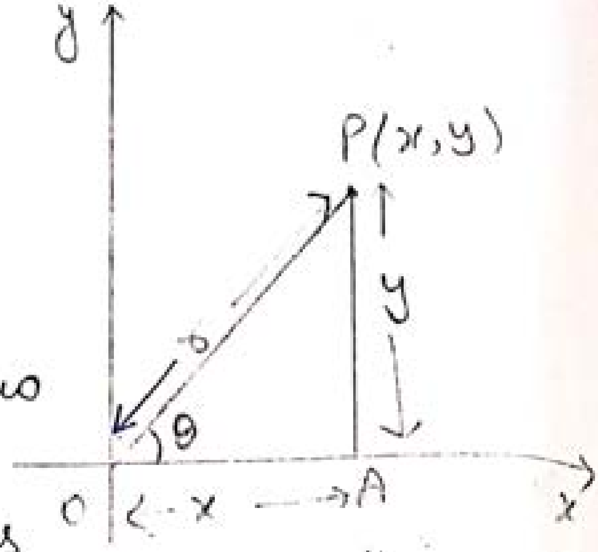
(v) If $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary.

(vi) If $z - \bar{z} = 0 \Rightarrow z$ is purely real.

(vii) $z.\bar{z} = x^2 + y^2 = |\bar{z}|^2$

$= |z|^2$

Polar standard form (Trigonometric form) (or modulus-argument form) :-



$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$z = re^{i\theta}$$

Hence, every non-zero complex no. $(x+iy)$ can be represented as polar form $r(\cos\theta + i\sin\theta)$ where, r and θ both are real numbers. Then,

$$|z| = r = \sqrt{x^2+y^2}$$

$$\text{argument } z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$\sin\theta = \frac{y}{r}$

$\Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r}$

$\Rightarrow x = r\cos\theta$

The value of argument which satisfy the condition $-\pi < \theta < \pi$ is called principle value of argument.

$|z| = |re^{i\theta}|$
$= |r||e^{i\theta}|$
$= |r| = r$

If $x, y > 0$, θ lies in Ist qua.

$x, y < 0$, θ lies in IIIrd qua.

$x > 0, y < 0$, θ lies in IVth qua.

$x < 0, y > 0$, θ lies in IInd qua.

Properties of modulus and argument :-

Thm :- If z is any non-zero complex no. then
$|\bar{z}| = |z|$ and $\arg z = -\arg \bar{z}$

Proof :- Let $z = r(\cos\theta + i\sin\theta)$ s.t. —— (1)
$|z| = r$ and $\theta = \arg z$

Now, $\bar{z} = r(\cos\theta - i\sin\theta)$
$= r(\cos(-\theta) + i\sin(-\theta))$

which is polar form of $\bar{z}$.

$\therefore$ $|\bar{z}| = r$, $\arg\bar{z} = -\theta$ —— (2)

from (1) & (2), we get
$|z| = |\bar{z}|$ and $\arg\bar{z} = -\arg z$ **Proved.**

Thm :- If z_1, z_2 are any two complex no.'s s.t. $z_1 \neq 0$ and $z_2 \neq 0$

Then, $|z_1 z_2| = |z_1| . |z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Proof :- Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ s.t.

$|z_1| = r_1$, $|z_2| = r_2$, $\arg z_1 = \theta_1$,
$\arg z_2 = \theta_2$ —— (1)

Now, $z_1 z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2 . (\cos\theta_2 + i\sin\theta_2)]$

$= r_1 r_2 [\cos\theta_1 \cos\theta_2 + i\sin\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i^2 \sin\theta_1 . \sin\theta_2]$

$= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i$
$(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$

$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

which is standard form of $z_1 z_2$

$\therefore$ $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 . z_2) = \theta_1 + \theta_2$

Using (1)

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

<u>Proved.</u>

<u>Thm</u> :- If z_1, z_2 are any two non-zero complex numbers. Then,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$ and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 .$$

<u>Proof</u>:- Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$|z_1| = r_1, |z_2| = r_2, \arg z_1 = \theta_1, \arg z_2 = \theta_2$ ⟵ (1)

Now,

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

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$$= \frac{r_1\left[\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2\right]}{r_2(\cos^2\theta_2 - i^2\sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

which is the standard form of $\frac{z_1}{z_2}$.

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

Using (1)

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proved

Thm :-

Triangle Inequality :-

If z_1, z_2 be two complex numbers. Then

$|z_1 + z_2| \le |z_1| + |z_2|$

or

The modulus of the sum of two complex numbers never exceed the sum of their modulus.

Proof :- The inequality is obviously true if one of z_1 or z_2 is zero.

Let $z_2 = 0 \Rightarrow |z_2| = 0$

Then,

$|z_1+z_2| = |z_1+0| = |z_1|+0 = |z_1|+|z_2|$

$\therefore$ it is true that $|z_1+z_2| \le |z_1|+|z_2|$

if $z_2 = 0$.

Now, let us assume that

$z_1 \neq 0$ and $z_2 \neq 0$ such that

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1+z_2 = (r_1(\cos\theta_1) + r_2(\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

Now

$$|z_1+z_2| = \sqrt{\left[(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2\right]}$$

$$= \sqrt{\cancel{r_1^2+r_2^2+2}}$$

$$= \sqrt{r_1^2(\cos^2\theta_1+\sin^2\theta_1) + r_2^2(\cos^2\theta_2+\sin^2\theta_2) + 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}$$

$$|z_1+z_2| = \sqrt{r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)}$$

but $\cos(\theta_1-\theta_2) \le 1$

$$\Rightarrow |z_1+z_2| \le \sqrt{r_1^2+r_2^2+2r_1r_2}$$

$$\le \sqrt{(r_1+r_2)^2} = (r_1+r_2)$$

$$\Rightarrow |z_1+z_2| \le |z_1|+|z_2| \quad \text{Proved}$$

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Parallelogram Property :-

If z_1, z_2 are two complex no.'s then

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$$

Proof :- Since, $|z|^2 = z \cdot \bar{z}$

Then,

$$|z_1+z_2|^2 + |z_1-z_2|^2$$

$$= (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2})$$

$$= (z_1+z_2)(\bar{z_1}+\bar{z_2}) + (z_1-z_2)(\bar{z_1}-\bar{z_2})$$

$$= z_1\bar{z_1} + \cancel{z_1\bar{z_2}} + \cancel{z_2\bar{z_1}} + z_2\bar{z_2} + z_1\bar{z_1} - \cancel{z_1\bar{z_2}}$$
$$- \cancel{z_2\bar{z_1}} + z_2\bar{z_2}$$

$$= 2z_1\bar{z_1} + 2z_2\bar{z_2}$$

$$= 2|z_1|^2 + 2|z_2|^2$$

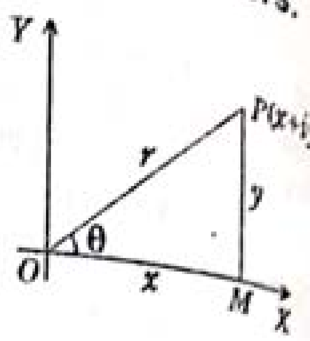
$$= 2\{|z_1|^2 + |z_2|^2\}$$ Proved.

... Representation of Algebric ...

+ive , it lies between

§ 9. The Geometrical Representation Of Complex Numbers.

Argand Diagram. A complex number $z = x + iy$ can be represented by a point P in the cartesian plane whose coordinates are (x, y) referred to rectangular axes OX and OY, usually called the **real** and **imaginary** axes respectively.

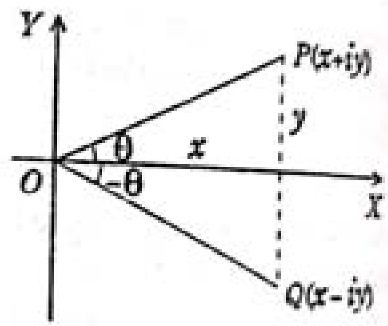


The complex number $0 + i0$ corresponds to the origin, the real numbers $x = x + i0$ correspond to the points on the x-axis and the purely imaginary numbers $iy = 0 + iy$ correspond to the points on the y-axis.

Obviously the polar coordinates of the point P are (r, θ) where $r = OP = \sqrt{(x^2 + y^2)}$ is the modulus and $\theta = \angle POX = \tan^{-1}(y/x)$ is the argument of the complex number z.

Thus θ is the angle made by OP with positive direction of x-axis. This representation of complex numbers as points in the plane is due to Argand and is called the **Argand diagram** or **Argand plane** or **Complex plane.**

The complex number z is known as the **affix** of the point (x, y) which represents it.



If two complex numbers z_1 and z_2 are represented in the Argand diagram, then from the definitions of the difference of two complex numbers and the modulus of a complex number it is obvious that $|z_1 - z_2|$ is the distance between the points z_1 and z_2. It follows that for a fixed complex number z_0 and a given +ive real number r, the equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r.

If the complex number $z = x + iy$ is represented by the point $P(x, y)$ in the Argand plane, then its conjugate $\bar{z} = x - iy$ is represented by the point $Q(x, -y)$ which is the image of the point P in the **real axis** OX. If (r, θ) are the polar coordinates of P, then the polar coordinates of Q are $(r, -\theta)$ so that we have $|z| = |\bar{z}|$ and $\arg z = -\arg \bar{z}$.

Thus if the trigonometrical representation of a complex number z is $r(\cos\theta + i\sin\theta)$, then that of $\bar{z}$ is

$$r\{\cos(-\theta) + i\sin(-\theta)\} \text{ i.e., } r(\cos\theta - i\sin\theta).$$

Using exponential form if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.

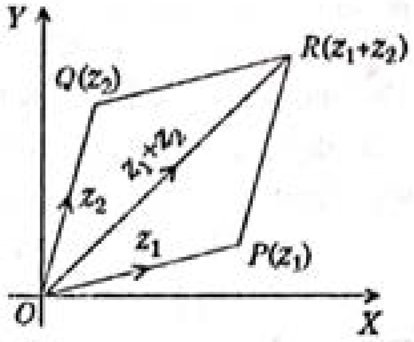
Vector representation of a complex number. If we represent a complex number $z = x + iy$ by a point P in the Argand plane, then the length of the line segment OP is equal to the modulus of the complex number z and the direction of OP is represented by $\arg z$. Therefore the complex number z can be represented by the vector OP and we write $z = \overrightarrow{OP}$.

§ 10. The Points On The Argand Plane Representing The Sum, Difference, Product And Division Of Two Complex Numbers.

(i) Representation of $z_1 + z_2$. Let the complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

be represented by the points P and Q on the Argand diagram. Then the coordinates of P and Q are (x_1, y_1) and (x_2, y_2) respectively. Complete the parallelogram $OPRQ$. Then the middle points of PQ and OR are the same. But the middle point of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which is therefore also the middle point of OR and so the coordinates of R are $(x_1 + x_2, y_1 + y_2)$. Thus the point R corresponds to the complex number



$$(x_1 + x_2) + i(y_1 + y_2)$$

i.e., $$(x_1 + iy_1) + (x_2 + iy_2) \quad i.e., \quad z_1 + z_2.$$

Therefore the sum $z_1 + z_2$ of the complex numbers z_1, z_2 is geometrically represented by the vertex R of the parallelogram $OPRQ$ whose adjacent sides OP and OQ are represented by the complex numbers z_1 and z_2.

The modulus and argument of $z_1 + z_2$ are given by

$$|z_1 + z_2| = OR \quad \text{and} \quad \arg(z_1 + z_2) = \angle ROX.$$

In vector notation, we have

$$z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}.$$

To deduce that $|z_1 + z_2| \leq |z_1| + |z_2|$.

We know that in any triangle, the sum of any two sides is greater than the third side. Therefore from ΔOPR, we have

$OR \leq OP + PR,$ the equality sign being also taken because the points O, P and R may be collinear

or $$OR \leq OP + OQ$$

or $$|z_1 + z_2| \leq |z_1| + |z_2|.$$

(ii) Representation of $z_1 - z_2$. Let the complex numbers

$$z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$$

be represented by the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively. Produce QO to Q' such that $OQ' = OQ$. Then the coordinates of the point Q' are $(-x_2, -y_2)$ and so the point Q' represents the complex number $-x_2 - iy_2$ i.e., $-z_2$.

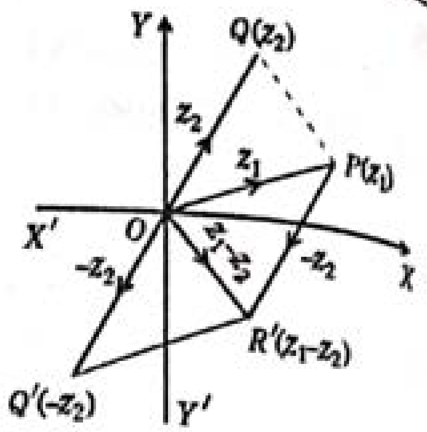
Complete the parallelogram $OQ'R'P$.

Then $$-z_2 = \overrightarrow{OQ'}$$

and $$z_1 - z_2 = z_1 + (-z_2) = \overrightarrow{OP} + \overrightarrow{OQ'} = \overrightarrow{OP} + \overrightarrow{PR'} = \overrightarrow{OR'}.$$

Thus the complex number $z_1 - z_2$ is geometrically represented by the vertex R' of the parallelogram $OQ'R'P$.

Since OQ is equal and parallel to $R'P$, therefore $OR'PQ$ is also a parallelogram and so $\overrightarrow{OR'} = \overrightarrow{QP}$. Thus the complex number $z_1 - z_2$ is also represented by the vector $\overrightarrow{QP}$.



We have $|z_1 - z_2| = OR' = QP$,

and $\arg(z_1 - z_2) = \angle R'OX$

i.e., the angle through which OX has to rotate so as to be in the direction of QP.

To deduce that $|z_1 - z_2| \geq |z_1| - |z_2|$.

We know that in a triangle the difference of any two sides is less than the third side. Therefore from ΔOPQ, we have

$$OP - OQ \leq QP$$

or $\quad |z_1| - |z_2| \leq |z_1 - z_2| \quad$ or $\quad |z_1 - z_2| \geq |z_1| - |z_2|$.

Remark. (a) Obviously $|z_1 - z_2| = QP$ and $\arg(z_1 - z_2)$ is the angle through which OX has to rotate in anti-clokwise direction as to be parallel to line QP. It is often convenient to use the polar representation about some point z_0 other than the origin. The representation $z - z_0 = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$ means that ρ is the distance between z and z_0 *i.e.* $\rho = |z - z_0|$, and ϕ is the angle of inclination of vector $z - z_0$ with the real axis. Further if the vector $z - z_0$ is rotated about z_0 in the anti-clockwise direction through an angle θ and z' is the new position of z, then

$$z' - z_0 = \rho e^{i(\phi + \theta)} = \rho e^{i\phi} . e^{i\theta} = (z - z_0)\, e^{i\theta}. \quad \text{(Note)}$$

(b) Let the lines AB and CD intersect at the point P_0 represented by the complex number z_0 and let P_1, P_2 be any two points on AB and CD represented by z_1 and z_2 respectively. Then the angle θ between the lines is given by

$$\theta = \arg(z_2 - z_0) - \arg(z_1 - z_0) = \arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right).$$

[Note that here only principal values of the arguments are considered].

If AB coincides with CD, then $\arg\{(z_2 - z_0)/(z_1 - z_0)\} = 0$ or π so that $(z_2 - z_0)/(z_1 - z_0)$ is real. It follows that the points A, B, C, D are collinear.

If AB is perpendicular to CD, then

$$\arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \pm\frac{\pi}{2} \text{ and so } \frac{z_2 - z_0}{z_1 - z_0} \text{ is pure imaginary.}$$

(iii) Representation of $z_1 z_2$ and z_1 / z_2. Let P and Q be the points corresponding to the complex numbers z_1 and z_2, where

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then $\quad OP = |z_1| = r_1, \; OQ = |z_2| = r_2,$

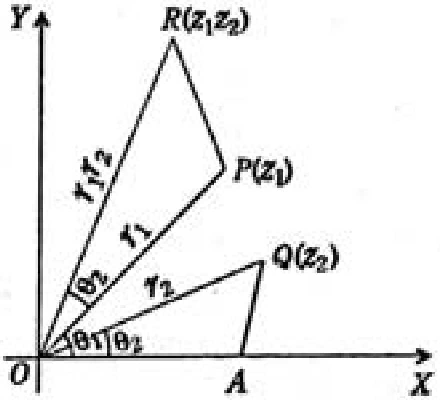
and $\quad \angle POX = \arg z_1 = \theta_1, \; \angle QOX = \arg z_2 = \theta_2.$

Representation of $z_1 z_2$. We have

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$$
$$+ i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$$
$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

From this representation of $z_1 z_2$ in standard polar form, we observe that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$



., *the modulus of the product of two complex numbers equal to the product of their moduli ;*

d $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

., *the argument of the product of two complex numbers equal to the sum of their arguments.*

Now the point R in the Argand diagram presenting the complex number $z_1 z_2$ can be obtained the following manner.

Take the point A on OX such that $OA = 1$.

Draw the triangle OPR similar to the triangle OAQ such that the points R and lie on the opposite sides of OP,

$$\angle ROP = \angle QOA = \theta_2$$

d
$$\angle OPR = \angle OAQ.$$

Then
$$\angle ROX = \angle POX + \angle ROP = \theta_1 + \theta_2$$
$$= \arg(z_1 z_2).$$

Also from similar triangles OAQ and OPR, we have

$$\frac{OR}{OQ} = \frac{OP}{OA}$$

$$OR = \frac{OP \,.\, OQ}{OA} = OP \,.\, OQ \qquad [\because\ OA = 1]$$

$$OR = r_1 r_2 = |z_1||z_2| = |z_1 z_2|.$$

Thus OR is the modulus of the complex number $z_1 z_2$ and $\angle ROX$ is the argument $z_1 z_2$. Hence the product $z_1 z_2$ is represented in the Argand diagram by the point

Remark. Multiplication by i.

Let $z = r(\cos\theta + i\sin\theta)$.

Since $i = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi$,

refore $zi = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$.

Hence multiplication of z with i rotates the vector for z through a right angle in positive direction.

Representation of z_1/z_2. We have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

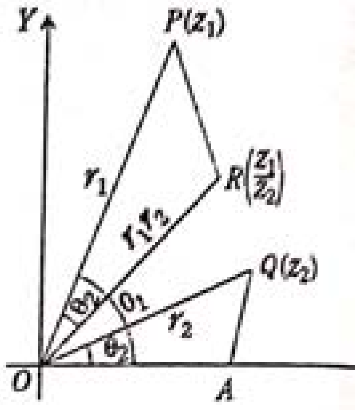
From the representation of z_1/z_2 in standard polar form, we observe that

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

i.e., the modulus of the quotient of two complex numbers is equal to the quotient of their moduli,

and $\quad \arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

i.e., the argument of the quotient of two complex numbers is equal to the difference of their arguments.



Now the point R in the Argand diagram representing the complex number z_1/z_2 can be obtained in the following manner :

Take the point A on OX such that $OA = 1$. Draw the triangle ORP similar to triangle OAQ such that the points R and Q are on the same side of OP,

$$\angle POR = \angle QOA = \theta_2$$

and $\quad \angle OPR = \angle OQA.$

Then $\angle ROX = \angle POX - \angle POR = \theta_1 - \theta_2 = \arg(z_1/z_2)$.

Also from similar triangles OAQ and ORP, we have

$$\frac{OR}{OA} = \frac{OP}{OQ} \text{ or } OR = \frac{OA \cdot OP}{OQ} = \frac{OP}{OQ} \qquad [\because OA =$$

or $\quad OR = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|.$

Thus OR is the modulus of the complex number z_1/z_2 and $\angle ROX$ is the argum of z_1/z_2.

Hence the quotient z_1/z_2 is represented in the argand diagram by the point R.

Remark. We have proved that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

and $\quad \arg(z_1/z_2) = \arg z_1 - \arg z_2.$

But if $\arg z_1$ and $\arg z_2$ are the principal values of the arguments, the $\arg z_1 + \arg z_2$ need not represent the principal value of the argument of $z_1 z_2$. A simil remark applies to

$$\arg z_1 - \arg z_2.$$

For example, if

$$z_1 = -1 + i,\ z_2 = 1 + i\sqrt{3},$$

$$\arg z_1 = \frac{3}{4}\pi,\ \arg z_2 = \frac{1}{3}\pi,$$

hen

$$\arg z_1 + \arg z_2 = \frac{3}{4}\pi + \frac{1}{3}\pi = \frac{13}{12}\pi > \pi.$$

o that

Therefore, $\arg z_1 + \arg z_2$ cannot be the principal value of $\arg (z_1 z_2)$.

11. More Properties Of Moduli And Arguments.

Theorem 1. *The modulus of the product of any number of complex numbers equal to the product of their moduli and the argument of the product is equal to the um of their arguments.*

Proof. Let $z_1, z_2, \ldots\ldots, z_n$ be the complex numbers. Let $r_1, r_2, \ldots\ldots, r_n$ denote eir moduli and $\theta_1, \theta_2, \ldots\ldots, \theta_n$ their arguments, so that

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1), \ldots\ldots, z_n = r_n (\cos \theta_n + i \sin \theta_n).$$

We have $z_1 z_2 \ldots\ldots z_n = \{r_1 (\cos \theta_1 + i \sin \theta_1)\}\{r_2 (\cos \theta_2 + i \sin \theta_2)\} \ldots\ldots \ldots\ldots\{r_n (\cos \theta_n + i \sin \theta_n)\}$

$$= r_1 r_2 \ldots\ldots r_n [\cos (\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots\ldots + \theta_n)]$$

$\therefore$ $$|z_1 z_2 \ldots\ldots z_n| = r_1 r_2 \ldots\ldots r_n = |z_1||z_2| \ldots\ldots |z_n|,$$

nd $$\arg (z_1 z_2 \ldots\ldots z_n) = \theta_1 + \theta_2 + \ldots\ldots + \theta_n$$

$$= \arg z_1 + \arg z_2 + \ldots\ldots + \arg z_n.$$

INEQUALITIES OF MODULI

Theorem 2. *The modulus of the sum of two complex numbers can never xceed the sum of their moduli.*

Proof. Let z_1, z_2 be any two complex numbers. We are to prove that

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

We know that $|z|^2 = z\bar{z}$.

$\therefore$ $$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1 \overline{z_1} + z_2 \overline{z_2} + (z_1 \overline{z_2} + z_2 \overline{z_1}).$$

Now $z_1 \overline{z_1} = |z_1|^2,\ z_2 \overline{z_2} = |z_2|^2.$

Also $z_1 \overline{z_2} + z_2 \overline{z_1} = z_1 \overline{z_2} + \overline{(z_1 \overline{z_2})}$

$$= 2R(z_1 \overline{z_2}). \qquad [\because\ z + \bar{z} = 2R(z)]$$

$\therefore$ $$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2R(z_1 \overline{z_2})$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1 \overline{z_2}| \qquad [\because\ R(z) \le |z|]$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||\overline{z_2}| \qquad [\because\ |z| = |\bar{z}|]$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$= [|z_1| + |z_2|]^2.$$

Thus $|z_1 + z_2|^2 \le [|z_1| + |z_2|]^2$

$\therefore$ $$|z_1 + z_2| \le |z_1| + |z_2|.$$

1] Funcⁿ of a Complex Variable :-

A complex funcⁿ is a funcⁿ in which the independent variable (z) and dependent variable (w) are both complex no. Therefore, A complex variable 'w' is said to be a funcⁿ of complex variable z if for each value of

$$z = x + iy$$

in region Rather correspond one or more definite values

region - R
domain - D
contour - C

of w. Then,

$$z = x + iy$$

$$w = f(z) = f(x+iy)$$

$$= u + iv$$

i.e. $u = f(x,y)$ & $v = f(x,y)$

Single Valued And Multi Valued func^n :-

If w takes only one value for each value of z in region R. Then, w is said to be a single valued func^n of z in R.

If there correspond two or more values of w for some or all values of z in region R. Then, w is said to be a multi valued func^n of 'z' in R.

Continuity of a Complex Variable :- (Same as real)

A f^n $w = f(z)$ is said to be cts at $z = z_0$ if $f(z_0)$ exist and $\lim_{z \to z_0} f(z)$ exist and $\lim_{z \to z_0} f(z) = f(z_0)$.

Further, $f(z)$ is said to be cts in any regio R of the z-plane (argand plane), if it is cts at every pt of R.

Diffrentiable:- A func^n $\omega = f(z)$ defined in region R is said to be diff^r at a pt $z = z_0$ if the limit

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ exist.}$$

If $\omega = f(z)$ is diff^r in R then the derivation of $f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}.$$

Analytic func^n :- A single valued func^n $\omega = f(z)$ in region R is said to be analytic at a pt $z = a$ if $f(z)$ is diff^r not only a but at every pt of some nbd of a.

An analytic func^n is also known as regular f^n or holomorphic f^n or monogenic func^n.

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2

Analytic function (Holomorphic / Regular function / Monogenic)

We can talk of analyticity of a function when the function is single valued

i) Analyticity of $w = f(z)$ at a point z_0

if $w = f(z)$ is diff^able at z_0 and also diff^able in some neighbourhood of z_0

ii) Analyticity of $w = f(z)$ in a region R

if $w = f(z)$ is diff^able at each point of R.

Harmonic func^n :- Any func^n of x and y having its partial derivatives of first and second order and also satisfying laplace eq^n is called harmonic func^n.

If $u = u(x, y)$ then u_x, u_y, u_{xx} and u_{yy} exist and u is harmonic if

what about u_{xy} & u_{yx}.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Entire func^n :- If $w = f(z)$ is analytic at every value of z except at ∞ ~~in region R~~, then $f(z)$ is called entire func^n.

Properties of Analytic f^n :-

If $f(z)$ and $g(z)$ be two analytic f^n, then

(i) $f+g$ (ii) $f-g$ (iii) $f \cdot g$ (iv) $k \cdot f$ where, k is any scalar. (v) f/g, provided $g(z) \neq 0$.

All are analytic in region R.

Cauchy-Riemann Eq^n :- (it is necessary condition for analytic fun^n)

इस को जाँच से analytic fun^n है
की सम्भावना है नहीं भी।

(i) **Cartizion plane :-**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$u_x = v_y \quad \& \quad u_y = -v_x$$

(ii) **Polar plane :-**

$$w = f(z)$$

and $z = r(\cos\theta + i\sin\theta)$

$$= re^{i\theta}$$

$$u + iv = f(re^{i\theta})$$

on partial derivatives w. r. to 'r'

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = e^{i\theta} \cdot f'(re^{i\theta}) \quad \text{---(1)}$$

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Again, partial derivatives w.r. to 'θ'

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r(i)e^{i\theta} f'(re^{i\theta}) \quad \text{—— (2)}$$

from (1) & (2)

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = r(i)\cdot\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\frac{\partial u}{\partial r} + i^2 r\frac{\partial v}{\partial r}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

On comparing

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}}$$

Ques-1. Express in the form of $A+iB$

(i) $\dfrac{2+3i}{4+5i}$, (ii) $\dfrac{3-2i}{7+4i}$, (iii) $\dfrac{1}{(1-2i)(2+3i)}$

Qs-2. Find modulus and argument

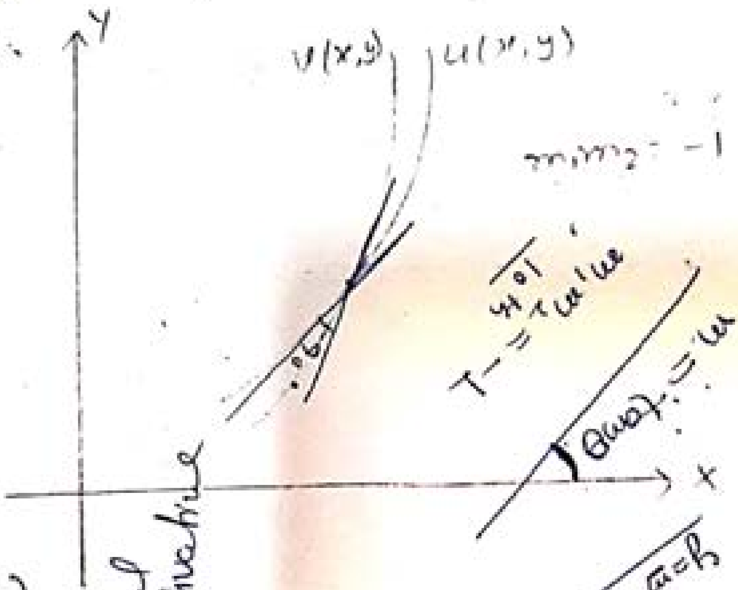
(i) $\dfrac{(2+i)^2}{(3-i)^2}$, (ii) $\dfrac{(2+i)}{4i+(i+1)^2}$,

(iii) $\dfrac{1-i}{1+i}$, (iv) $\dfrac{3-i}{2+i}+\dfrac{3+i}{2-i}$

Qs-3. If $x+iy = \dfrac{3}{2+\cos\theta + i\sin\theta}$, prove that

$(x-1)(x-3)+y^2=0$.

Theorem:- When the func^n $f(z) = u+iv$ is analytic ~~Sol~~. then prove that the families of curves $u(x,y)=C_1$ and $v(x,y)=C_2$ are orthogonal. (Intersect at $90°$)



Proof:-

Let $f(z) = u+iv$ is analytic func^n such that

$u(x,y) = C_1$ ——(1)

$v(x,y) = C_2$ ——(2)

Total derivative

$$① \Rightarrow \boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}dx = -\frac{\partial u}{\partial y}dy$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (let)} \quad \text{——(3)}$$

$$(3) \Rightarrow dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2 \text{ (let)} \quad \text{———} (4)$$

Since, $f(z)$ is analytic.

∴ from cauchy-riemann eqn.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \& \quad \text{—(i)} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \quad \text{—(ii)} \end{aligned}\right\} \text{———} (5)$$

Now, u and v are orthogonal, if

$$m_1 m_2 =$$

$$\text{Now, } m_1 m_2 = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

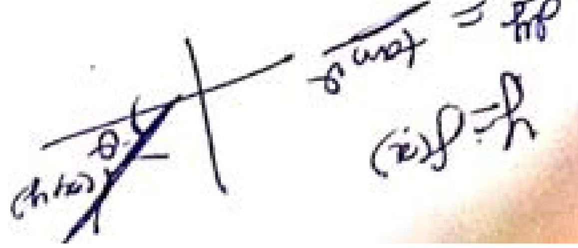
(ii) (i)

$$= \frac{\cancel{\frac{\partial u}{\partial x}}}{-\cancel{\frac{\partial v}{\partial x}}} \cdot \frac{\cancel{\frac{\partial v}{\partial x}}}{\cancel{\frac{\partial u}{\partial x}}}$$

$$m_1 m_2 = -1$$

Hence, u and v are orthogonal.

Proved.



Ques-1. Express in the form of $A+iB$.

(i) $$\frac{2+3i}{4+5i} = \frac{(2+3i)}{4+5i} \times \frac{(4-5i)}{4-5i} = \frac{8-10i+12i-15i^2}{16-25i^2}$$

$$= \frac{23+2i}{41} = \frac{23}{41} + i\frac{2}{41} \quad \underline{\text{Ans.}}$$

(ii) $$\frac{3-2i}{7+4i} = \frac{(3-2i)(7-4i)}{(7+4i)(7-4i)}$$

$$= \frac{21-12i-14i+8i^2}{49-16i^2} = \frac{13-26i}{65}$$

$$= \frac{1}{5} - \frac{2}{5}i \quad \underline{\text{Ans.}}$$

(iii) $$\frac{1}{(1-2i)(2+3i)} = \frac{1}{2+3i-4i-6i^2}$$

$$= \frac{1}{(8-i)} = \frac{(8+i)}{8-i^2}$$

$$= \frac{8+i}{9} = \frac{8}{9} + i.\frac{1}{9} \quad \underline{\text{Ans.}}$$

Ques-2. Find modulus and argument

(i) $$z = \frac{(2+i)^2}{(3-i)^2} = \frac{4+i^2+4i}{9+i^2-6i} = \frac{3+4i}{8-i6} \times \frac{8+6i}{8+6i}$$

$$= \frac{24+18i+32i+24i^2}{64-36i^2}$$

$$= \frac{0+50i}{100}$$

$$z = \frac{50}{100}i = \frac{1}{2}i$$

modulus $|z| = \sqrt{\frac{1}{4}+0}$

$$= \frac{1}{2}$$

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argument, $\theta = \tan^{-1}(y/x)$

$= \tan^{-1}(h/0)$

$= \tan^{-1}(\infty) = \frac{\pi}{2}$ Ans.

(ii) $\frac{2+i}{4i+(i+1)^2} = \frac{2+i}{4i+i^2+1+2i}$

$= \frac{2+i}{6i} \times \frac{i}{i}$

$= \frac{2i+i^2}{6i^2} = \frac{-1}{-6} - \frac{2i}{6}$

$= \frac{1}{6} - \frac{1}{3}i$

$|z| = \sqrt{\frac{1}{36} + \frac{1}{9}}$

$= \sqrt{\frac{1+4}{36}} = \frac{\sqrt{5}}{6}$

$\theta = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{-\frac{1}{3}}{\frac{1}{6}}\right)$

$\theta = \tan^{-1}(-2)$ Ans.

(iii) $\frac{1-i}{1+i} = \frac{(1-i)^2}{1-i^2}$

$= \frac{1+i^2-2i}{2} = -i$

$|z| = \sqrt{1} = 1$

$\theta = \tan^{-1}(-1/0) = \frac{\pi}{2}$ Ans.

(iv) $\frac{3-i}{2+i} + \frac{3+i}{2-i}$

$$= \frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$$

$$= \frac{6-3i-2i+i^2+6+3i+2i+i^2}{4-i^2}$$

$$= \frac{4}{3}$$

$$|z| = \frac{4}{3}$$

$$\theta = \tan^{-1}\left(\frac{0}{4/3}\right) = \tan^{-1}(0) = 0 \quad \underline{\text{Ans.}}$$

Qs-3. $x+iy = \frac{3}{2+\cos\theta+i\sin\theta}$, prove that

$$(x-1)(x-3)+y^2=0$$

$$\Rightarrow \quad x+iy = \frac{3}{(2+\cos\theta)+i\sin\theta} \times \frac{(2+\cos\theta)-i\sin\theta}{(2+\cos\theta)-i\sin\theta}$$

$$\Rightarrow \quad (x+iy) = \frac{6+3\cos\theta-i3\sin\theta}{(2+\cos\theta)^2-i^2\sin^2\theta}$$

$$= \frac{(6+3\cos\theta)-3i\sin\theta}{4+\cos^2\theta+4\cos\theta+\sin^2\theta}$$

$$x+iy = \frac{(6+3\cos\theta)-3i\sin\theta}{5+4\cos\theta}$$

on comparing,

$$x = \frac{6+3\cos\theta}{5+4\cos\theta}, \quad y = \frac{-3\sin\theta}{5+4\cos\theta}$$

L.H.S.

$$(x-1)(x-3)+y^2 = 0$$

$$\Rightarrow \left(\frac{6+3\cos\theta}{5+4\cos\theta}-1\right)\left(\frac{6+3\cos\theta}{5+4\cos\theta}-3\right)+\left(\frac{-3\sin\theta}{5+4\cos\theta}\right)^2$$

$$\Rightarrow \frac{(1-\cos\theta)(-9-9\cos\theta)}{(5+4\cos\theta)^2}+\frac{9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$\Rightarrow \frac{-9(1+\cos\theta)(1-\cos\theta)+9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$\Rightarrow \frac{-9(1-\cos^2\theta)+9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$\Rightarrow \frac{-9+9\cos^2\theta+9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$\Rightarrow \frac{-9+9}{(5+4\cos\theta)^2} = 0 = \text{R.H.S.}$$

Proved

Theorem:-1 (3/2)

The necessary condition for $w = f(z)$ to be analytic at any point $z = x+iy$ is that ~~u_x, u_y, v_x and v_y should exist and~~ the function satisfies Cauchy-riemann eqns

Proof:- Let $f(z)$ be analytic at any point z in a region R. then by differentiation, we have

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z)-f(z)}{\delta z} \quad \text{--- (1)}$$

Now, $f(z) = u + iv$

$\therefore f(z+\delta z) = (u+\delta u) + i(v+\delta v)$ ——— (2)

Now using (2) eq^n in (1), we have

$$f'(z) = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{\delta u + i\delta v}{\delta z}$$

$$= \lim_{\delta z \to 0} \left[\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right] \quad ——— (3)$$

Since, $f'(z)$ exist and the lim along two path must be equal.

(i) First along x-axis :-

$y = 0 \Rightarrow \delta y = 0$

$\because z = x + iy$

$\delta z = \delta x + i\delta y$

$\delta z = \delta x$

notation change delta

$$③ \Rightarrow f'(z) = \lim_{\delta x \to 0} \left[\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right]$$

del

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad ——— (4)$$

$\delta z = \delta x + i\delta y$

(ii) Along y-axis :- $x = 0 \Rightarrow \delta x = 0$

$\Rightarrow \delta z = i\delta y$

$$\therefore ③ \Rightarrow f'(z) = \lim_{\delta y \to 0} \left[\frac{\delta u}{i\delta y} + \frac{i\delta v}{i\delta y}\right]$$

$$= \lim_{\delta y \to 0} \left[\frac{\delta v}{\delta y} + (-i)\frac{\delta u}{\delta y}\right]$$

$$\Rightarrow \quad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{——— (5)}$$

from (4) & (5), we have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

On comparing real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

&

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \left[\begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}\right]$$

which are known as Cauchy-Riemann eqⁿs.

<u>Proved.</u>

<u>Theorem</u>:- The real and imaginary parts u and v of an analytic funcⁿ $f(z)$ are harmonic.

<u>Proof</u>:- Let $f(z)$ be an analytic funcⁿ in region* R. Then, Cauchy Riemann eqⁿs are satisfied.

∴ we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——— (2)}$$

On partially diffⁿ (1) w.r. to x. and eqⁿ (2) w.r. to y. and adding both eqⁿ.

$(1) \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x},$

$(2) \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$

$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Hence, u is harmonic func^n.

Now, (1) partially diff. w.r. to 'y' and (2) w.r. to 'x' and subtracting, we have

$(1) \Rightarrow \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2},$

$(2) \Rightarrow \quad \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2}$

$\Rightarrow 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$

Hence, v is harmonic func^n. Proved.

• Laplace Eq^n in polar form :-

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Theorem :- Sufficient Condition for the func^n $w = f(z)$ is analytic in region R if

Th-2 (310)

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$

(ii) u_x, u_y, v_x, v_y all are cts in region R.

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Th. 2 (Proof) U_y $\frac{\partial U}{\partial y}$ (3)

$$f(z) = u + iv$$
$$= u(x,y) + iv(x,y)$$

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$w = f(z) = u(x,y) + iv(x,y)$$

$u = u(x,y)$ so $u + \delta u = u(x+\delta x, y+\delta y)$

$$u(x,y) + \delta u = u(x+\delta x, y+\delta y)$$

$$\delta u = u(x+\delta x, y+\delta y) - u(x,y)$$

$$\delta u = \underbrace{u(x+\delta x, y+\delta y) - u(x+\delta x, y)} + \underbrace{u(x+\delta x, y) - u(x,y)}$$

$$\delta u = \delta y\, U_y(x+\delta x, y+\theta_1\delta y) + \delta x\, U_x(x+\theta_2\delta x, y) \quad (1)$$

$0 < \theta_1 < 1$ & $0 < \theta_2 < 1$ by Lagrange mean value theorem.

Lagrange's Mean value theorem: If f be defined and continuous on $[a, a+h]$ and diff.able $]a, a+h[$, then $\exists$ a point $c = a + \theta h$ $(0 < \theta < 1)$ in the open interval $]a, a+h[$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(a+\theta h)$$

$$f(a+h) - f(a) = h f'(a+\theta h)$$

change

Since U_x & U_y are continuous in the given region,

$\therefore$ by def. of ~~uniform~~ continuity, we have

$$|u_y(x+\delta x, y+\theta_1 \delta y) - u_y(x,y)| < \epsilon_1$$

$$\& \ |u_x(x+\theta_2 \delta x, y) - u_x(x,y)| < \epsilon_2 \quad \text{(2)}$$

ϵ is very small the value

whenever $|\delta y| < \eta_1$ & $|\delta x| < \eta_2$ respective

Let $u_y(x+\delta x, y+\theta_1 \delta y) - u_y(x,y) = \alpha_1$

& $u_x(x+\theta_2 \delta x, y) - u_x(x,y) = \alpha_2$ (3)

Then from (2) $|\alpha_1| < \epsilon_1$ & $|\alpha_2| < \epsilon_2$

putting ~~these~~ values from (3) in (1)

$$\delta u = \delta y\{\alpha_1 + u_y(x,y)\} + \delta x\{\alpha_2 + u_x(x,y)\}$$

Similarly,

$$\delta v = \delta y\{\beta_1 + v_y(x,y)\} + \delta x\{\beta_2 + v_y(x,y)\}$$

where $|\alpha_1| < \epsilon_1$ & $|\alpha_2| < \epsilon_2$ & $|\beta_1| < \epsilon_3$ & $|\beta_2| < \epsilon_4$

$a \to$ con f^n give $\epsilon > 0$ $\exists \ \delta > 0$

$|f(x) - f(a)| < \epsilon$ if $|x-a| < \delta$

Now $\cancel{\dfrac{\delta f}{\delta z} = \dfrac{\delta u + i\delta v}{\delta x + i\delta y} = u_y \delta}$

$$\delta u = u_y(x,y)\,\delta y + \alpha_1 \delta y + u_x(x,y)\,\delta x + \alpha_2 \delta x$$

$$\delta u = \frac{\partial u}{\partial y}\delta y + \alpha_1 \delta y + \frac{\partial u}{\partial x}\delta x + \alpha_2 \delta x$$

$$\delta u = \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial x}\delta x + \underline{\alpha_1}\delta y + \underline{\alpha_2}\delta x$$

$\alpha_1 \longrightarrow 0$, $\alpha_2 \longrightarrow 0$ as $\delta y \to$ & $\delta x \longrightarrow 0$

Similarily

$$\delta v = \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial x}\delta x + \beta_1 \delta y + \beta_2 \delta x$$

$\beta_1 \longrightarrow 0$, $\beta_2 \longrightarrow 0$ as $\delta y \longrightarrow 0$, $\delta x \longrightarrow 0$

$(w = f(z))$ $\quad \underline{f(z + \delta z) = \cancel{\delta w}\, w + \delta w}$

$\cancel{\delta w}$

$$\delta w = \delta u + i \delta v$$

$$= \Big(\frac{\partial u}{\partial x}\underbrace{\delta x}_{+\alpha_1\delta y + \alpha_2 \delta x} + \frac{\partial u}{\partial y}\delta y\Big) + i\Big(\frac{\partial v}{\partial x}\underbrace{\delta x}_{+\beta_1\delta y + \beta_2 \delta x} + \frac{\partial v}{\partial y}\delta y\Big)$$

$$= \Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big)\delta x + \Big(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\Big)\delta y + (\alpha_2 + i\beta_2)\delta x + (\alpha_1 + i\beta_1)\delta y$$

$$\begin{bmatrix} u_x = v_y \\ u_y = -v_x \end{bmatrix}$$

$$= \Big(\frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}\Big)\delta x + \Big(\frac{-\partial v}{\partial x} + \frac{i\partial u}{\partial x}\Big)\delta y$$

$$= \Big(\frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}\Big)\delta x + \Big(\frac{-\partial v}{i\partial x} + \frac{\partial u}{\partial x}\Big) i\,\delta y$$

$$= \left(\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta x + \left(i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) i\delta y + \text{''} + \text{''}$$

$$= \underline{\left(\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)}\delta x + \left(\underline{\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}}\right) i\delta y + \text{''} + \text{''}$$

$$\delta w = \left(\frac{\partial v}{\partial x} + \frac{i\partial u}{\partial x}\right)(\delta x + i\delta y) + (\epsilon_2 + i\beta_2)\delta x + (\epsilon_1 + i\beta_1)\delta y$$

Then on dividing by $\delta z = \delta x + i\delta y$
& taking limit as $\delta z \to 0$, $\delta x + i\delta y \to 0$

$\delta x \to 0$
$\delta y \to 0$

$$\boxed{\frac{dw}{dz}} = f'(z) = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} =$$

$\frac{\partial u(x,y)}{\partial x}$ ~~$\frac{\partial}{\partial}$ $\frac{\partial f}{\partial}$~~

$$= \lim_{\delta z \to 0} \frac{\left(\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)(\delta x + i\delta y)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$$

$$f'(z) \quad \boxed{\frac{dw}{dz} = \frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}}$$

derivative of $f(z)$ exists, function $w = f(z)$ is analytic

$$f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \delta w$$

Construction of Analytic Function :-

Let $f(z) = u + iv$ be an analytic f^n where both u and v are harmonic conjugate func^{n}s of each other. If one of these is given then to find other.

Method (1) :-

Let $f(z) = u + iv$ be an analytic func^n and u is given. we have to find v.

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

but $f(z)$ is analytic.

$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore \quad dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{——— (1)}$$

Eqⁿ (1) can be written as

$$dv = M\,dx + N\,dy$$

when, $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

Now $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

But u is harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

which is condition of exact diffⁿ eqⁿ.

So, eqⁿ (1) can be integrated.

$$\Rightarrow \int dv = \int M\,dx + \int N\,dy + c$$

$$\Rightarrow v = \int M\,dx + \int N\,dy + c$$

(treat y as constant) (Neglect terms of x) ?

Q^n Show that the following functions are harmonic and find their harmonic conjugate functions.

i) $u = \frac{1}{2}\log(x^2+y^2)$ ii) $v = \sinh x \cos y$

Sol^n

(ii) $v = \sinh x \cos y$

$\frac{\partial v}{\partial x} = \cosh x \cos y \Rightarrow \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y$

$\frac{\partial v}{\partial y} = -\sinh x \sin y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y$

Since, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ $\therefore$ v is harmonic

now $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$ (using C-R eq^ns)

$= -\sinh x \sin y\, dx - \cosh x \cos y\, dy$

$= -[\sinh x \sin y\, dx + \cosh x \cos y\, dy]$

$du = -d(\cosh x \sin y)$

integration yields

$u = -\cosh x \sin y + c$ which is the required harmonic conjugate of v.

March-19 / Sam. $\frac{\partial u}{} = \frac{\partial v}{}$ and $\frac{\partial u}{}$ ...

$$\left.\begin{aligned} v &= \sinh x \cos y \\ u &= ? \end{aligned}\right] \quad f(z) = u + iv$$

$$\frac{\partial v}{\partial x} = \cosh x \cos y \Rightarrow \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y$$

v is harmonic since $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \frac{\partial v}{\partial y} dx + \left(-\frac{\partial v}{\partial x}\right) dy \quad \text{(C-R eqns)}$$

$$= (-\sinh x \sin y) dx - \cosh x \cos y \, dy$$

$$du = -[\sinh x \sin y \, dx + \cosh x \cos y \, dy]$$

$$du = -d[\cosh x \sin y]$$

integrating

$$u = -\cosh x \sin y + C$$

Method (2) :- (Milne Thomson method):-

Let u is given and v is

to be find out.

$$z = x+iy \text{ and } \bar{z} = x-iy$$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i} \qquad \text{——(1)}$$

Now,

$$w = f(z) = u(x,y) + iv(x,y)$$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2}\right)$$

$$\text{——(2)}$$

Now put, $z = \bar{z}$ in eq^n (2)

$$f(z) = u(z,0) + iv(z,0)$$

that gives $x = z$ and $y = 0$

Now

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (\text{using C-R eq}^n\text{s})$$

Let $\frac{\partial u}{\partial x} = \phi_1(x,y) = \phi_1(z,0)$

$$\frac{\partial u}{\partial y} = \phi_2(x,y) = \phi_2(z,0)$$

$$\Rightarrow \quad f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

on int.

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

* If v is given and u is to be find out

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$$

where, $\phi_1(z,0) = \frac{\partial v}{\partial y}$, $\phi_2(z,0) = \frac{\partial v}{\partial x}$

Milne Thompson Method (U given)

$z = x+iy$ add $z+\bar{z} = 2x \Rightarrow x = \frac{1}{2}(z+\bar{z})$

$\bar{z} = x-iy$ subtract $z-\bar{z} = 2iy \Rightarrow y = \frac{1}{2i}(z-\bar{z})$

$f(z) = u+iv$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = u\left(\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ —— (1)

$z = x+iy$

put, $\bar{z} = z$.

$x-iy = x+iy$

$y=0$

$z = x+iy$

$\boxed{z = x}$

$f(z) = u(z,0) + iv(z,0)$ —— (2)

U given

Consider, $f(z) = u+iv$

$f'(z) = u_x + iv_x$

$= u_x + i(-u_y)$

$f'(z) = u_x - iu_y$.

f^n of x & y

$u_x = \phi_1(x,y)$ & $u_y = \phi_2(x,y)$

$f'(z) = \phi_1(x,y) - i\phi_2(x,y)$

$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$ $\left[\begin{matrix} x = z \\ y = 0 \end{matrix}\right]$

integrate

$f(z) = \int \{\phi_1(z,0) - i\phi_2(z,0)\} dz + c$

constant real or complex

where $\phi_1(x,y) = \frac{\partial u}{\partial x}$, $\phi_2(x,y) = \frac{\partial u}{\partial y}$

---

v given

$f(z) = \int \{\psi_1(z,0) + i\psi_2(z,0)\} dz + c$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ & $\psi_2(x,y) = \frac{\partial v}{\partial x}$

Milne Thompson Method

Extended

$u - v$ is given

$f(z) = u + iv$ —①

$if(z) = iu - v$ —②

①+②

$\underbrace{(1+i)f(z)}_{F(z)} = \underbrace{(u-v)}_{U} + i\underbrace{(u+v)}_{V}$

$F(z) = U + iV$

$(1+i)f(z) = F(z)$

$$\frac{d\left(\frac{1}{q}\right)}{} \quad \frac{q\,dp - p\,dq}{q^2}$$

(b)

$$F(z) = \int \{ \psi_1(t,0) + i\,\psi_2(t,0)\,dt + C$$

$$\psi_1 = \frac{\partial V}{\partial y}, \qquad \psi_2 = \frac{\partial V}{\partial x}$$

$$V = \frac{x}{x^2+y^2}$$

$$= V_x = \frac{\partial V}{\partial x} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(\;\;)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{-z^2}{z^4} \quad \frac{-1}{z^2}$$

$$V_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$\psi_1(z,0)$$

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Ques. Show that the func^n

$$f(z) = (x^2+y) + i(y^2-x)$$ is everywhere cts but not analytic.

Solu^n:- $f(z) = u+iv$

$$\therefore \quad u = x^2+y, \quad v = y^2-x$$

$$u_x = 2x, \quad u_y = 1, \quad v_x = -1, \quad v_y = 2y$$

$\Rightarrow$ $f(z)$ is cts everywhere

$$u_x \neq v_y, \quad u_y = -v_x$$

$\Rightarrow$ C.R. eq^{n}s are not satisfied.

$\Rightarrow$ $f(z)$ is not analytic. Ans-

Ques- Prove that $f(z)$ is entire f^n and find its derivaties where,

$$f(z) = e^x \cos y + i e^x \sin y$$

Solu^n:- $f(z) = u+iv$

$$\therefore \quad u = e^x \cos y, \quad v = e^x \sin y$$

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y,$$

$$v_y = e^x \cos y$$ each of these partial derivatives ~~is~~ can be diffed to find the derivatives, so these are continuous.

Here, $u_x = v_y$ and

$$u_y = -v_x$$

Since C-R eq^{n}s holds

$\therefore$ $f(z)$ is ~~cts~~ analytic.

$\Rightarrow$ $f(z)$ is entire f^n.

★ **Note :-** To show f^n is entire it is enough that f^n is analytic (C-R eq^ns) hold.

$f(z) = u + iv$

Now, $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

क्या इसे w.r.t. to y निकाल सकते हैं?

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x . e^{iy}$$

$$f'(z) = e^{x+iy} \quad \text{Ans-}$$

Ques Show that the f^n $f(z) = z.|z|$ is not analytic.

Soln:-

$$z = x + iy$$

$$|z| = \sqrt{x^2+y^2}$$

$$\therefore z|z| = (x+iy)\sqrt{x^2+y^2}$$

$$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$$

$$\therefore u = x\sqrt{x^2+y^2}, \quad v = y\sqrt{x^2+y^2}$$

$$u_x = \sqrt{x^2+y^2} + x\frac{1 . \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, \quad v_x = \frac{y . \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}, \quad = \frac{xy}{\sqrt{x^2+y^2}}$$

$$u_y = \frac{x . \cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}, \quad v_y = \sqrt{x^2+y^2} + \frac{y . \cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$\Rightarrow u_x \neq v_y, \quad u_y \neq -v_x$

$\Rightarrow$ C.R. eq^ns are not satisfied.

$\Rightarrow f(z)$ is not analytic. Ans-

Que! - Find the values of A and B such that the func^n

$$f(z) = x^2 + Ay^2 - 2xy + i(Bx^2 - y^2 + 2xy)$$ is analytic. Also find its derivative.

Solu^n:- Here,

$$u = x^2 + Ay^2 - 2xy \quad \&$$

$$v = Bx^2 - y^2 + 2xy$$

Since, f(z) is analytic.

∴ from C-R eq^{n}s, we have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$2x - 2y = -2y + 2x \quad \& \quad 2Ay - 2x = -[2Bx + 2y]$$

$$\Rightarrow 2Ay - 2x = -2Bx - 2y$$

On comparing the coeff. of x and y, we get

$$-2 = -2B \Rightarrow \boxed{B = 1}$$

$$\& \quad -2 = 2A$$

$$\Rightarrow \boxed{A = -1}$$

Now,

$$f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (2x - 2y) + i(2x + 2y)$$

$$= 2x - 2y + 2ix + 2iy$$

$$= 2(x + iy) + 2(ix - y)$$

$= 2(x+iy) + 2(ix+i^2y)$

$= 2(x+iy) + 2i(x+iy)$

$f'(z) = 2z + 2iz$ Ans-

Que:- If ϕ and ψ are func^{n}s of x and y, satisfing laplace eq^n then show that $(s+it)$ is analytic. where, $s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}$,

$$t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}$$

Proof:- Let,

$$f = s + it$$

To show f is analytic, we show that C-R eq^{n}s are satisfied for s and t. i.e.

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \quad \& \quad \frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \quad \text{---(1)}$$

By hypothesis ϕ and ψ satisfy laplace eq^{n}s

i.e. $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{---(2)}$$

Now, $$\frac{\partial s}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2},$$

$$\frac{\partial s}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\& \quad \frac{\partial t}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y},$$

$$\frac{\partial t}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2}$$

Now,

$$\frac{\partial s}{\partial x} - \frac{\partial t}{\partial y}$$

$$= \cancel{\frac{\partial^2 \phi}{\partial x \partial y}} - \frac{\partial^2 \psi}{\partial x^2} - \left(\cancel{\frac{\partial^2 \phi}{\partial x \partial y}} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$= -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial s}{\partial x} - \frac{\partial t}{\partial y} = 0 \Rightarrow \boxed{\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}}$$

Again,

$$\frac{\partial s}{\partial y} + \frac{\partial t}{\partial x}$$

$$= \frac{\partial^2 \phi}{\partial y^2} - \cancel{\frac{\partial^2 \psi}{\partial y \partial x}} + \frac{\partial^2 \phi}{\partial x^2} + \cancel{\frac{\partial^2 \psi}{\partial x \partial y}}$$

$$= \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}$$

$$\Rightarrow \frac{\partial s}{\partial y} + \frac{\partial t}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}}$$

Since, s and t satisfied Laplace - Eq^n.

$\therefore$ $s + it$ is Analytic. Ans.

Ques :-

$\phi + i\psi \rightarrow$ stream func^n

$\downarrow$ velocity potential

In two dimensional fluid flow the

stream funcⁿ

$\psi = \dfrac{-y}{x^2+y^2}$. find velocity potential (ϕ)

for fluid flow.

Solⁿ:- Let $G = \phi + i\psi$

where, $\psi = -\dfrac{y}{x^2+y^2}$ (imaginary part)

Now, $\psi_1(z,0) = \dfrac{\partial\psi}{\partial y} = \dfrac{-[(x^2+y^2)(+1) - y\cdot(2y)]}{(x^2+y^2)^2}$

$= \dfrac{-(x^2-y^2)}{(x^2+y^2)^2}$

Put $x = z$ & $y = 0$

$\psi_1(z,0) = \dfrac{-z^2}{z^4} = -\dfrac{1}{z^2}$

$\psi_2(z,0) = \dfrac{\partial\psi}{\partial x} = \dfrac{+y\cdot 2x}{(x^2+y^2)^2}$

$\psi_2(z,0) = 0$

By milne - Thomson method, we have

$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$

$f(z) = \int -\dfrac{1}{z^2}\,dz + c$

$= -\int z^{-2}\,dz + c$

$f(z) = -\left(\dfrac{z^{-1}}{-1}\right) + c$

$$\boxed{f(z) = \frac{1}{z} + c}$$

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Put $z = x+iy$

$$f(z) = \frac{1}{x+iy} + c$$

$$\phi + i\psi = \frac{x-iy}{x^2+y^2} + c$$

$$\Rightarrow \phi + i\psi = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} + c$$

$$\Rightarrow \boxed{\phi = \frac{x}{x^2+y^2} + c}$$ Ans-

Ques- Find analytic funcⁿ if v is given by

$$v = y^2 - x^2$$

Solⁿ:- $\psi_1(z,0) = \frac{\partial v}{\partial y} = 2y$

$$\psi_1(z,0) = 0$$

$$\psi_2(z,0) = \frac{\partial v}{\partial x} = -2x$$

$$\psi_2(z,0) = -2z$$

By, milne ~~theore~~ Thomson method, we get

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c$$

$$= 0 + i\int -2z\,dz + c$$

$$= -\cancel{2}i.\frac{z^2}{\cancel{2}} + c$$

$$\boxed{f(z) = -iz^2 + c}$$ Ans.

Put $z = x + iy$

$$f(z) = -i(x+iy)^2 + c$$
$$= -i(x^2 - y^2 + 2xiy) + c$$
$$= -ix^2 + iy^2 - 2xyi^2 + c$$
$$= -ix^2 + iy^2 + 2xy + c$$
$$u + iv = 2xy + i(y^2 - x^2) + c$$
$$\Rightarrow u = 2xy + c, \quad v = y^2 - x^2 \quad \text{Ans.}$$

Ques → Verify Laplace Eqⁿ if
$$u = \left(r + \frac{a^2}{r}\right)\cos\theta$$

Ques → Show that the f^n e^z is entire f^n.

Ques → Determine constant b such that $u = e^{bx}\cos 5y$ is harmonic funcⁿ. And also find its harmonic conjugate.

Theorem :- Th-3 5Ⓐ Prove that an Analytic funcⁿ with constant modulus is constant.

Proof :- Let $f(z) = u + iv$ be an analytic funcⁿ with constant modulus. Then

$$|f(z)| = |u + iv| = \text{constant}$$
$$\Rightarrow \sqrt{u^2 + v^2} = c \text{ (say)}$$

if $c = 0$ $\quad u^2 + v^2 = 0$ which is possible only if $v = u = 0$
$f(z) = 0$. So let $c \neq 0$

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sq^2 both sides,

$$u^2 + v^2 = c^2 \quad \text{——(1)}$$

On. diff. w.r. to 'x', we get

$$2u \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \quad \text{——(2)}$$

Again, diff. (1) w.r. to 'y', we get

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0 \quad \text{——(3)}$$

$\because$ $f(z)$ is analytic.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$(3) \Rightarrow u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad \text{——(4)}$$

* sq^2 and adding eq^{n}s (2) & (4)

5 (1)

$$u^2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) + v^2\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) = 0$$

$$\Rightarrow (u^2 + v^2)\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) = 0$$

$$\because u^2 + v^2 = c^2 \neq 0$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad \left\{ \begin{array}{l} \because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \end{array} \right.$$

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$\Rightarrow |f'(z)|^2 = 0$

$\Rightarrow |f'(z)| = 0$

$\Rightarrow f(z)$ is constant. Proved Ans.

5 1

Th-3 $\underset{\text{analytic}}{f(z)}$ $|f(z)| = \text{constant}$ $|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$

(*) sq² and adding eqⁿ (3) & (4)

$$\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)^2 = 0$$

$$\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + \cancel{2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}\right]$$

$$+\left[u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - \cancel{2uv\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)}\right] = 0$$

$$u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] + v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\underbrace{(u^2+v^2)}_{c} = 0$$

$f(z) = u + iv$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$

Put

$\Rightarrow |f'(z)| = 0$

$\Rightarrow f'(z) = 0$

$\Rightarrow f(z) = c$

~~$z^2 = x+iy$~~

~~$z^2 = x^2 - y^2 + 2ixy$~~

~~$|z^2|$~~ $|z|^2$

$\Rightarrow$ f(z) is constant. Proved Ans.

Ques:- Verify Laplace Eq^n if

$$u = \left(r + \frac{a^2}{r}\right)\cos\theta. \quad \text{---(1)}$$

Sol^n:- On. diff. (1) w. r. to 'r'

$$\frac{\partial u}{\partial r} = \left[1 + a^2\left(-\frac{1}{r^2}\right)\right]\cos\theta.$$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

again, $$\frac{\partial^2 u}{\partial r^2} = +\frac{2a^2}{r^3}\cos\theta$$

And, on diff. (1) w. r. to 'θ'.

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta$$

again, $$\frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta.$$

We know that, Laplace eq^n in polar form is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

L.H.S. $$\frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{a^2}{r^2}\right)\cos\theta + \frac{1}{r^2}\left[-\left(r + \frac{a^2}{r}\right)\cos\right.$$

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$\Rightarrow \cancel{\frac{2a^2}{r^3}\cos\theta} + \cancel{\frac{1}{r}\cos\theta} - \cancel{\frac{a^2}{r^3}\cos\theta} - \cancel{\frac{1}{r}\cos\theta} - \cancel{\frac{a^2}{r^3}\cos\theta}$

$= 0$

$\Rightarrow$ Here, Laplace eq^n holds. Ans.

Qus :- Show that the func^n e^{z^2} is entire func^n.

Solu^n:- $f(z) = e^{z^2}$

$\because \quad z = x + iy$

$$\Rightarrow f(z) = e^{(x+iy)^2}$$

$$= e^{x^2-y^2+2ixy}$$

$$= e^{x^2-y^2} \cdot e^{i2xy}$$

$$= e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$$

$$f(z) = e^{x^2-y^2}\cdot\cos 2xy + i e^{x^2-y^2}\cdot\sin 2xy$$

Here, $u = e^{x^2-y^2}\cdot\cos 2xy$, $v = e^{x^2-y^2}\cdot\sin 2xy$

Now,

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}\cdot 2x\cdot\cos 2xy + e^{x^2-y^2}\cdot(-\sin 2xy\cdot 2y)$$

$$= e^{x^2-y^2}\cdot 2x\cos 2xy - e^{x^2-y^2}\cdot 2y\sin 2xy,$$

$$\frac{\partial u}{\partial y} = e^{x^2-y^2}\cdot(-2y)\cos 2xy + e^{x^2-y^2}\cdot(-\sin 2xy\cdot 2x)$$

$$= -e^{x^2-y^2}\cdot 2y\cos 2xy - e^{x^2-y^2}\cdot 2x\sin 2xy$$

And

$$\frac{\partial v}{\partial x} = e^{x^2-y^2} \cdot 2x \sin 2xy + e^{x^2-y^2} \cdot \cos 2xy \cdot 2y$$

$$\frac{\partial v}{\partial y} = e^{x^2-y^2} \cdot (-2y) \sin 2xy + e^{x^2-y^2} \cdot \cos 2xy \cdot 2x$$

u_x, u_y, v_x & v_y are continuously differentiable, so these fun^n are continuous.

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow$ C-R eq^{n}s holds.

$\therefore$ $f(z)$ is analytic.

$\Rightarrow$ $f(z) = e^{z^2}$ is entire func^n. Ans.

Que $\rightarrow$ Determine constant b such that $u = e^{bx} \cos 5y$ is harmonic func^n. And also find its harmonic conjugate.

Soln :- Since, u is harmonic.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Here, $u = e^{bx} \cos 5y$

$$\frac{\partial u}{\partial x} = b e^{bx} \cos 5y$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = b^2 e^{bx} \cos 5y$$

Again,

$$\frac{\partial u}{\partial y} = -5 e^{bx} \sin 5y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -25 e^{bx} \cos 5y$$

$(1) \Rightarrow \quad b^2 e^{bx}\cos 5y - 25 e^{bx}\cos 5y = 0$

$\Rightarrow \quad e^{bx}(b^2\cos 5y - 25\cos 5y) = 0$

$\Rightarrow \quad e^{bx}\cos 5y\,(b^2 - 25) = 0$

if $\quad b^2 - 25 = 0, \quad \because e^{bx}\cos 5y \neq 0.$

$\Rightarrow \quad \boxed{b = \pm 5}$

$\therefore \quad u = e^{\pm 5x}\cos 5y$

Now,

$\phi_1(z,0) = \frac{\partial u}{\partial x} = \pm 5 e^{\pm 5x}\cos 5y$

$\phi_1(z,0) = 5e^{5z}.1 = \pm 5 e^{\pm 5z}$

$\phi_2(z,0) = \frac{\partial u}{\partial y} = -5 e^{\pm 5x}\sin 5y.$

$\phi_2(z,0) = 0$

By. Milne thomson method, we get

$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz + c$

$= \int 5e^{\pm 5z}\,dz + 0 + c$

$= \cancel{5}.\frac{e^{\pm 5z}}{\cancel{5}} + c$

$\boxed{f(z) = e^{\pm 5z}}$

$= e^{\pm 5(x+iy)}$

$= e^{\pm 5x}.e^{\pm 5iy}$

$u + iv = e^{\pm 5x}(\cos 5y + i\sin 5y)$

$\Rightarrow \boxed{v = e^{\pm 5x}\sin 5y}$ Ans.

Ques:- Determine Analytic funcⁿ whose real part is given by

$$u = e^{-x}(x\cos y + y\sin y) \text{ and } f(0) = i$$

Solⁿ:-

$$\phi_1(z,0) = \frac{\partial u}{\partial x} = -e^{-x}(x\cos y + y\sin y) + e^{-x}(1.\cos y)$$

$$\phi_1(z,0) = -e^{-z}(z) + e^{-z}$$

$$= e^{-z}(1-z)$$

$$\phi_2(z,0) = \frac{\partial u}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y)$$

$$\phi_2(z,0) = e^{-z}(-z\sin 0 + 0\times\cos 0 + \sin 0)$$

$$= 0$$

By, MTM

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz + A$$

$$= \int (e^{-z} - ze^{-z})\,dz + A$$

$$= -e^{-z} - \left[z.\frac{e^{-z}}{-1} - \int 1.\frac{e^{-z}}{-1}\,dz\right] + A$$

$$= -\cancel{e^{-z}} + ze^{-z} + \cancel{e^{-z}} + A$$

$$\boxed{f(z) = ze^{-z} + A}$$

$\because$ $f(0) = i$

$f(0) = i = 0.e^{-0} + A$

$\Rightarrow$ $\boxed{A = i}$

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$\therefore$ $\boxed{f(z) = ze^{-z} + i}$ Ans.

Note.

Ques → Find Analytic funcⁿ if

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

Solⁿ :- $\phi_1(z,0) = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$

$$\phi_1(z,0) = 3z^2 + 6z$$

$$\phi_2(z,0) = \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\phi_2(z,0) = -6.z.0 - 0$$
$$= 0$$

By MLT.

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz + A$$

$$= \int (3z^2 + 6z)\,dz + A$$

$\boxed{f(z) = z^3 + 3z^2 + A}$ Ans.

Q. If $u = x^2 - y^2$, $v = \frac{y}{x^2+y^2}$ both u and v satisfy laplace's equation but $u+iv$ is not analytic function of z

OR

Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic functions of x and y but are not harmonic conjugates.

Sol^n $u = x^2 - y^2$ $\quad \frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -2$

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9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Hence $u(x,y)$ satisfies Laplace's eq^n and hence harmonic

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$$v = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

Again $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ $\therefore$ v is harmonic

But $u_x \neq v_y$ and ~~$v_x \neq$~~ $u_y \neq -v_x$

but $u + iv$ is not analytic.

(2)

...ch-9 - Sh: Ex - Give an example: a function which is differentiable but not analytic.

Qs. Ex. Show that the function $f(z) = |z|^2$ is not analytic anywhere.

Solution of above both Ex.

$$f(z) = |z|^2 = z\bar{z}$$

{ Analytic f^n is independent of $\bar{z}$ }

1. We prove the def. of analyticity of a f^n at a point

$w = f(z) = x^2 + y^2$

Consider origin $(0,0)$ z_0 be general point.

$\left(\frac{dw}{dz}\right)_{z=z_0} = f'(z_0)$

$z = x+iy$, $\bar{z} = x-iy$

$z\bar{z} = x^2+y^2$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$

$$= \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)\overline{(z_0+\Delta z)} - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\bar{z}_0+\overline{\Delta z}) - z_0\bar{z}_0}{\Delta z}$$

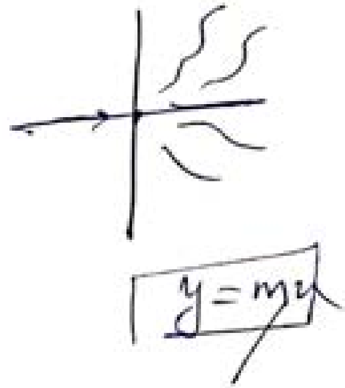
$$= \lim_{\Delta z \to 0} \frac{\{z_0\bar{z}_0 + z_0\overline{\Delta z} + \bar{z}_0\Delta z + \Delta z\overline{\Delta z}\} - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[z_0 \frac{\overline{\Delta z}}{\Delta z} + \bar{z}_0 + \overline{\Delta z} \right]$$

$$= \left[\lim_{\Delta z \to 0} z_0 \frac{\overline{\Delta z}}{\Delta z}\right] + (\bar{z}_0) + (0)$$

$\Delta z \to 0$
$(\Delta x + i\Delta y) \to 0$
$\Delta x \to 0$
$\Delta y \to 0$
$\Delta x - i\Delta y \to 0$
$\overline{\Delta z} \to 0$

$$\left(\frac{dw}{dz}\right)_{z=z_0} = f'(z_0) = \bar{z}_0 + \lim_{\Delta z \to 0} z_0 \frac{\overline{\Delta z}}{\Delta z}$$



Case(i) $z_0 = 0$

$$\left(\frac{dw}{dz}\right)_{z=0} = f'(0) = 0 + \lim_{\Delta z \to 0} 0 \, \frac{\overline{\Delta z}}{\Delta z}$$

$$\left(\frac{dw}{dz}\right)_{z=0} = f'(0) = 0$$

$f(z) = z\bar{z} = |z|^2$ is diff^{able} at origin (0,0)

Now

case(ii) $z_0 \neq 0$

$$\left(\frac{dw}{dz}\right)_{z=z_0\neq 0} = f'(z_0) = \bar{z}_0 + \lim_{\Delta z \to 0} z_0 \frac{\overline{\Delta z}}{\Delta z}$$

$$\left(\frac{dw}{dz}\right)_{z=z_0\neq 0} = \bar{z}_0 + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} z_0 \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

~~first~~ We find limit taking $\Delta y = m\Delta x$ as a path when m is a parameter

$$= \bar{z}_0 + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} z_0 \frac{\Delta x - im\Delta x}{\Delta x + im\Delta x}$$

$$= \bar{z}_0 + \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} z_0 \frac{(1-im)}{(1+im)}$$

$$= \bar{z}_0 + z_0 \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(1-im)}{(1+im)}$$

~~the value of the~~ limit depends on the parameter m, so the value of the limit is not unique

(X)

Thus $\left(\frac{dw}{dz}\right)_{z=z_0 \neq 0}$ does not exists

hence derivative of $f(z)$ does not exists

in the nbhd of origin.

$\boxed{\text{Continuity is the necessary condition for differentiability}}$

funcs dy

Ex-1 Kri-70 Krishna P = 70, Ex-2 funct.. dy f.

$u = \sqrt{|xy|}$, $x = 0$

$\sqrt{TT}$

$\left(\frac{\partial u}{\partial x}\right)_{x=a} = \lim_{x \to a} f(x$

~~$x^2 + y^2$~~

~~$\frac{\partial v}{\partial x} = 2x$~~

~~$\frac{\partial v}{\partial y} = 2y$~~

(a, b)

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h} \\ \frac{\partial f}{\partial y} &= \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k} \end{aligned}}$$

$(0,0)$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h, 0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$\frac{\partial u}{\partial x} = 0$

Similarly $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$

So, the C-R eq's are satisfied.

Ex-2 Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin although the C-R eq's are satisfied at the point. (C-R eq. of necessary condition.)

(3)

when to find at (0,0)

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z+z_0) - f(z_0)}{z}}$$

$z_0 = 0$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|xy|} - 0}{x+iy}$$

$y = mx$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|x^2 m|}}{x(1+im)}$$

$u_x = v_y$

$u_y = -v_x$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|m|}}{(1+im)}$$

thus limit does not unique at origin, since it depends on m. (in the neighbourhood of 0).

$\therefore f'(0)$ does not exist. then $f(z) = \sqrt{|xy|}$ is not analytic at origin.

(*) limit formula to find partial derivative u_x, u_y, v_x, v_y

: origin (0,0)

: f^n is complex to diff^{iate}

$z = 0 + i0$

$f(0) = 0$

$= u = 0, v = 0$

when $x = 0$, $y = 0$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

(6)

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)} \cdot \frac{1}{x + iy}$$

$y = mx$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3(1 - m^3) + i x^3(1 + m^3)}{x^2(1 + m^2)} \cdot \frac{1}{x(1 + im)}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

derivative doesn't exist